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# FREE-BOUNDARY PROBLEMS FOR NONLINEAR MODELS OF FLUID FILTRATION IN INHOMOGENEOUS POROUS MEDIA

**I. B. Davydkin**[1] **and V. N. Monakhov**[2]

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*Existence theorems are proved for solutions of problems of nonlinear gravity fluid filtration in regions with specified boundaries of complex geometry. The theory developed can be used to design the underground flow net of a hydraulic structure with specified filtration characteristics.*

**Key words:** *nonlinear filtration, free boundary, simple polygon, existence theorem.*

The nonlinear law of resistance of a porous medium to a fluid flowing through it (Darcy's law) was proposed by S. A. Khristianovich (1940), who established an analogy between the model obtained and the nonlinear subsonic gas-dynamic equations. Khristianovich's model has been widely used to describe oil motion in a porous bed. V. N. Monakhov (1961) was the first to prove existence theorems for solutions of problems of nonlinear filtration of a fluid with free boundaries using quasiconformal mapping. In the present paper, similar results are obtained for filtration regions with boundaries of specified complex geometry.

**1. Formulation of the Problem.** The physical aspects of the gravity filtration problem and results of previous studies are reported in monographs [1, 2] and review [3].

Steady-state nonlinear fluid filtration in an inhomogeneous porous medium (soil) is described by the following elliptic system of equations, whose solutions are obtained using quasiconformal mapping [4]:

$$-\boldsymbol{v} = K(z, \varphi, \nabla\varphi)\nabla\varphi, \qquad \operatorname{div}\boldsymbol{v} = 0 \qquad [\boldsymbol{v} = (v_1, v_2)]. \tag{1}$$

Here $K$ is the symmetric filtration tensor with components differentiable with respect to the argument, $z = x + iy$, $\xi = \varphi_x + i\varphi_y$, and $\varphi$ is the fluid potential (piezometric head). Setting $a_{ij} = \partial v_i/\partial\xi_j$ ($\xi_1 = \varphi_x$ and $\xi_2 = \varphi_y$) and $\alpha_{ij} = \int\limits_0^1 a_{ij}(z, \varphi, s\xi)\, ds$, we arrive at the representation $K = \{a_{ij}\}$ [4]. Here it is assumed that the quadratic form is $\Lambda(\xi, \lambda) = \sum\limits_{i,j=1}^{2} a_{ij}\lambda_i\lambda_j$ and, hence, $\Lambda_0(\xi, \lambda) = (K\lambda, \lambda) = \int\limits_0^1 \Lambda(s\xi, \lambda)\, ds$ are positively defined.

After introduction of the stream function $\psi(x, y) \equiv \psi(x_1, x_2)$, Eqs. (1) can be written as

$$\psi_y = \sum_{i=1}^{2} \alpha_{1i}\varphi_{x_i} = v_1, \qquad -\psi_x = \sum_{i=1}^{2} \alpha_{2i}\varphi_{x_i} = v_2$$

or in equivalent form for the function $z = z(w)$ ($w = \varphi + i\psi$):

$$z_{\bar{w}} - m_1(w, z, \sigma)z_w - m_2(w, z, \sigma)\bar{z}_{\bar{w}} = 0 \qquad (\sigma = z_w). \tag{2}$$

Here $m_i$ are explicitly expressed in terms of the components $\alpha_{ij}$ of the tensor $K$ continuously differentiable with respect to all arguments. According to [5], the equation for the function $z(w)$ is globally solvable for to $z_{\bar{w}}$, as is shown in the form of Eq. (2). Differentiation of both sides of relation (2) with respect to $w$ yields the following equation for $\sigma$:

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[1]Gorno-Altaisk State University, Gorno-Altaisk 659700. [2]Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 44, No. 6, pp. 76–84, November–December, 2003. Original article submitted March 4, 2003.

 

$$\sigma_{\bar{w}} - n_1(w,z,\sigma)\sigma_w - n_2(w,z,\sigma)\bar{\sigma}_{\bar{w}} = \sum_{k+l=0}^{2} b_{kl}\sigma^k\bar{\sigma}^l. \tag{3}$$

Here $n_i$ and $b_{kl}$ are expressed in terms of $m_i$ and their derivatives [4].

The assumptions of the differentiability of the tensor $K$ and the positive definiteness of $\Lambda(\xi,\lambda)$ and $\Lambda_0(\xi,\lambda)$ can be written as [4, 5]

$$\sup(\|m\|,\|n\|) < 1, \qquad \|b\| < \infty, \tag{4}$$

where $m = (m_1, m_2)$, $n = (n_1, n_2)$, and $b = \{b_{kl}\}$ are the vectors and matrix of the coefficients in Eqs. (2) and (3), respectively, $\|\varphi\| = \sup \sum_{k=1}^{s} |\varphi_k|$, and $\varphi = (\varphi_1, \ldots, \varphi_s)$. As is shown in [5], system (2)–(4) corresponds to the general nonlinear elliptic equation.

Let gravity fluid filtration occur in a domain $D$ bounded by a specified polygon $P$ with vertices $z_k$ and angles $\alpha_k\pi$ ($0 < \delta \leqslant \alpha_k \leqslant 2$ and $k = \overline{0, n+1}$) and an unknown curve $L$ — the boundary between the moistened and unmoistened parts of the porous medium. The specified polygon $P$ consists of impermeable domains $P^1$ and interfaces $P^2$ with the immovable fluid, which also include horizontal seepage segments. The desired complex filtration potential $w = \varphi + i\psi$ satisfies the following boundary conditions on $\partial D = P^1 \cup P^2 \cup L$:

$$\psi = \psi_1, \quad z \in P^1, \qquad \varphi = \varphi_1, \quad z \in P^2,$$

$$\varphi + x = \varphi_0 = \text{const}, \qquad \psi = \psi_0 = \text{const}, \qquad z \in L.$$

Here $\varphi_1$ and $\psi_1$ are the piecewise continuous functions on $P^1 \cup P^2$. The boundary conditions define the pre-image $D^*$ of the filtration domain $D$ [$D = z(D^*)$] under quasiconformal mapping of $z = z(w)$ by the solution of Eq. (2). In the case, the boundary $\partial D^*$ consists of segments of straight lines $\varphi = \text{const}$ and $\psi = \text{const}$.

Another example of this type of problems is the design of the flow net of a concrete hydraulic structure $L$ using a head profile or a flow rate distribution (drainage layer) specified on $L$:

$$\varphi = \varphi(x), \quad \psi = \text{const} \quad \text{if} \quad \varphi = \text{const}, \quad \psi = \psi(x).$$

Particular problems of this type in the class of analytic functions were solved for the first time by N. I. Kochina and P. Ya. Polubarinova-Kochina. Results of their studies are reported in [1, pp. 186–201]. The general problem of construction of an unknown segment $L$ of the boundary $D$ of the domain of definition of an analytic function $w(z) = \varphi + i\psi$ was formulated and solved by Monakhov [6, Chapter 3]:

$$G(\varphi,\psi) = 0, \quad z \in P, \qquad w = g(x), \quad z \in L. \tag{5}$$

In this case, as in the above-formulated problem of gravity fluid filtration and the problem of construction of a concrete dam contour, boundary conditions (5) define the image $D^*$ of the filtration domain $D$ [$D^* = w(D)$] in the plane of the complex potential $w = \varphi + i\psi$.

**2. Homogeneous Soil.** Steady-state fluid filtration in a homogeneous soil is described by an analytic function $w(z) = \varphi + i\psi$, which is a complex filtration potential. We construct a conformal map $w = W(\zeta)$, $W\colon E \to D^*$ of the upper half-plane $E$: $\operatorname{Im}\zeta > 0$ onto the specified domain $D^*$ whose boundary $\partial D^* = P^* \cup L^*$ is defined by boundary conditions (5).

2.1. *Differentiable Boundary Data.* Assumptions:

(i) The polygon $P \subset \partial D$ is simple [7];

(ii) The curves of $(P^*, L^*) \subset D^*$ defined by Eqs. (5) are Lyapunov's curves [$(P^*, L^*) \subset C^{1+\alpha}$, where $\alpha > 0$], and their points of intersection $w_0$ and $w_{n+1}$ have interior angles $\gamma_k\pi$ ($0 < \delta \leqslant \gamma_k \leqslant 2$ and $k = 0, n+1$).

If Assumptions (i) and (ii) are satisfied, according to [6, p. 110], the derivative $\omega \equiv dz/d\zeta$ of the conformal map $z\colon E \to D$ satisfies the boundary-value problem

$$\begin{aligned} &\operatorname{Re}[e^{i\pi(1/2-\delta_k)}\,\omega(t)] = 0, \qquad t \in [t_k, t_{k+1}], \\ &\operatorname{Re}\omega(t) = h(t) \equiv \Pi_0(t)h_*(t), \qquad l\colon |t| > 1. \end{aligned} \tag{6}$$

Here $\delta_k\pi$ is the slope of the $k$th side of the polygon $P$, $t_k$ are pre-images of the vertices $z_k \in P$, $t_0 = -1 < t_1 < \ldots < t_{n+1} = 1$, $\Pi_0(t) = \prod_{k=0,n+1} (t - t_k)^{\gamma_k - 1}$, $\ln h_*(t) \in C^\alpha(l)$ ($\alpha > 0$), $\alpha_k - \gamma_k < 1 - \delta$ ($k = 0, n+1$), and $\delta > 0$ is a characteristic of the simple polygon.

The solution of problem (6) is written as

$$\omega = \frac{dz}{d\zeta} = \frac{\Pi(\zeta)}{\pi i} \int\limits_{|t|>1} \frac{h(t)\,dt}{\Pi(t)(t-\zeta)} \equiv \Pi(\zeta)M(\zeta), \tag{7}$$

where $\Pi = \prod\limits_{k=0}^{n+1} (\zeta - t_k)^{\alpha_k - 1}$ and $\alpha_k\pi$ is the interior angles at the vertices $z_k \in P$ [6, p. 111]. The unknown variables $t_k$ $(k = \overline{1,n})$ are obtained from the equations that define the geometry of the polygon $P$:

$$l_k = |z_{k-1} - z_k| = \int\limits_{t_{k-1}}^{t_k} |\Pi(t)||M(t)|\,dt, \qquad k = \overline{1,n}.$$

As is found in [6], the parameters $t_k$ are determined uniquely and fit the inequalities

$$|t_{k+1} - t_k| > \varepsilon > 0, \qquad k = \overline{0,n}, \tag{8}$$

where $\varepsilon = \varepsilon(\|h_*\|^{(\alpha)}, \delta)$ ($\delta > 0$ is a constant in the definition of a simple polygon [7]); $\|\varphi\|^{(\alpha)} = \|\varphi\|_\Omega^{(\alpha)} = \|\varphi\|_{C^\alpha(\Omega)}$.

We introduce the weight function $\Pi_* = \prod\limits_{k=0}^{n+1} (\zeta - t_k)^{\beta_k}$. Here $\beta_k = 0$ for $\alpha_k \geqslant 1$ and $\beta_k = 1 - \alpha_k$ for $\alpha_k < 1$, where $k = \overline{1,n}$. If $k = 0, n+1$, then $\beta_k = 0$ for $1 < \alpha_k < \gamma_k$ and $\alpha_k \geqslant \gamma_k \geqslant 1$; $\beta_k = 1 - \alpha_k$ for $\alpha_k < \gamma_k \leqslant 1$ and $\alpha_k \leqslant 1 \leqslant \gamma_k$; and $\beta_k = 1 - \gamma_k$ for $\alpha_k \geqslant 1 \geqslant \gamma_k$. Then, the representation (7) leads to the estimate

$$\|\Pi_* z_\zeta\|_E^{(\nu)} = C(\varepsilon) < \infty, \qquad \nu = \nu(\alpha, \alpha_k) > 0. \tag{9}$$

2.2. *Boundary Conditions from $C^\alpha$.* Let us consider the following boundary-value problem for the function $z(\zeta)$:

$$z(s) = P, \quad s\colon |t| < 1, \qquad \operatorname{Re} z = H(t), \quad l\colon |t| > 1, \tag{10}$$

where $H(t) \in C^\alpha(l)$ $(\alpha > 0)$.

We choose functions $H_m(t) \in C^{1+\alpha}(l)$ such that $\|H_m - H\|_l^{(\alpha)} \to 0$ as $m \to \infty$ and set $h_m = dH_m/dt \in C^\alpha(l)$.

The derivative $dz_m/d\zeta$ of the conformal map $z_m\colon E \to D_m$ satisfies the boundary-value problem of the form of (6) [with $h = h_m(t)$, $|t| > 1$] and is represented as (7). Then, we have

$$z = \int\limits_{-1}^{\zeta} \Pi_m(\zeta) M_m(\zeta)\,d\zeta \equiv F_m(\zeta), \tag{11}$$

where $\Pi_m = \prod\limits_{k=0}^{n+1} (\zeta - t_k^m)^{\alpha_k - 1}$ and $M_m = M(\zeta)$ for $h\Pi^{-1} = h_m \Pi_m^{-1}$, and the desired constants $t_k^m$ satisfy the inequalities

$$|t_{k+1}^m - t_k^m| > \varepsilon_m > 0, \qquad k = \overline{0,n}. \tag{12}$$

We note that if the estimates (12) are satisfied uniformly with respect to $m$ $(\varepsilon_m \geqslant \varepsilon_0 > 0)$, the first condition in (10) can be written as

$$\operatorname{Re}[e^{i\pi(1/2 - \delta_k)}(z - z_k)] = 0, \qquad t \in [t_k, t_{k+1}].$$

Solving the boundary-value problem

$$\operatorname{Im} F_m = f_m(t), \quad |t| < 1, \qquad \operatorname{Re} F_m = H_m(t), \quad |t| > 1$$

and setting $f_m(t) \equiv 0$ $(|t| > 1)$ and $H_m(t) \equiv 0$ $(|t| < 1)$, we write the function $F_m(\zeta)$ from Eq. (11) in the form

$$F_m = B(H_m + i f_m|\zeta), \qquad B(\varphi|\zeta) = \frac{\sqrt{1-\zeta^2}}{\pi i} \int\limits_{-\infty}^{+\infty} \frac{\varphi(t)\,dt}{\sqrt{1-t^2}(t-\zeta)}. \tag{13}$$

According to the properties of the Cauchy-type integral $B(\varphi|\zeta)$, in Eq. (13), we have

$$\|B(H_m|t)\|_l^{(\alpha_0)} \leqslant C(\|H_m\|_l^{(\alpha_0)}) \leqslant C_0(\|H\|_l^{(\alpha_0)}) < \infty, \qquad \alpha_0 = \min(1/2, \alpha) > 0.$$

At the same time,

$$|B(if_m|t)| \leqslant |F_m| \leqslant \sum_{k=0}^{n} |z_{k+1} - z_k| = |P|, \qquad |t| < 1,$$

$$|B(if_m|t)| \leqslant \max_{\tau} |f_m(\tau)| \frac{\sqrt{1-t^2}}{\pi i} \int_{-1}^{1} \frac{d\tau}{\sqrt{1-\tau^2}(\tau - t)} = \max_{\tau} |f_m(\tau)| \leqslant |P|, \quad t < -1.$$

A similar estimate holds for $t > 1$. Therefore, for the analytic functions $F_m(\zeta)$ in the form of (13), the following uniform estimate holds:

$$|F_m(t)| \leqslant |P| + C_0, \qquad t \in (-\infty, \infty). \tag{14}$$

Obviously, this estimate holds for the entire domain $E$: $\operatorname{Im}\zeta > 0$.

From the compact [by virtue of (14)] sequence of the analytic functions $\{F_m(\zeta)\}$, we distinguish a subsequence $\{F_{m_k}(\zeta)\}$ that is uniformly convergent for $\operatorname{Im}\zeta > 0$. It is easy to see that the limit function $z = F_0(\zeta)$ satisfies the boundary-value problem (10) [6, pp. 130–131].

For any function $\varphi(\zeta) \in W_p^1(E)$ $(p > 1)$, we designate its trace on $\Gamma \subset \partial E$ by $\varphi(t) \in SW_p^1(\Gamma)$ and set $Q_\rho = \{\zeta\colon |\operatorname{Re}\zeta| > 1 + \rho, \operatorname{Im}\zeta > \rho\}$. Thus, we proved the following theorem:

**Theorem 1.** *There exists at least one solution $z = F_0(\zeta)$ of the boundary-value problem* (10) *for analytic functions that satisfies the estimate* (14).

*If $H(t) \in SW_{p>2}^1(l)$ or $H(t) \in C^\alpha(l)$ $(\alpha > 0)$, then, $z(\zeta) \in W_{p_0>2}^1(Q_\rho)$ or $z(\zeta) \in C^{\alpha_0}(Q_\rho)$, respectively, $p_0 = p_0(p, \delta)$, and $\alpha_0 = \alpha_0(\alpha, \delta)$, where $\delta > 0$ is a characteristic of the simple polygon $P$.*

**3. Inhomogeneous Soil.** Let the coefficients in Eq. (2) do not depend on $\sigma = z_w$, which corresponds to the case of an inhomogeneous soil under quasilinear Darcy's law, i.e., in (1), $K = K(z, \varphi)$.

In [8], the solvability of the problem (2), (10) was proved and interior estimates for the quasiconformal map $z$: $D^* \to D$ were derived. In the present paper, we will also obtain some estimates up to the boundary. As in Sec. 2, we construct a conformal map $w = W(\zeta)$, $W$: $E \to D^*$ of the upper half-plane $E$ onto the domain $D^*$. By virtue of Assumption (ii), $(P^*, L^*) \subset C^{1+\alpha}$ $(\alpha > 0)$, and, hence,

$$\frac{dW}{d\zeta} = \prod_{k=0,n+1} (\zeta - t_k)^{\gamma_k - 1} R(\zeta), \qquad \ln R \in C^\alpha(E).$$

Then, Eq. (2) reduces to

$$z_{\bar\zeta} - \mu_1 z_\zeta - \mu_2 \bar z_{\bar\zeta} = 0, \qquad \|\mu\| < 1, \tag{15}$$

where $\mu_1(\zeta, z) = m_1 \bar w_{\bar\zeta}/w_\zeta$, $\mu_2(\zeta, z) = m_2$, and $\mu = (\mu_1, \mu_2)$.

Let $z = F(\zeta)$, $F$: $E \to D$ be the desired quasiconformal map of Eq. (15). We substitute an arbitrary measurable function $z^0(\zeta)$ into the coefficients of (15) and consider a quasiconformal map $\xi = \xi(\zeta)$, $\xi$: $E \to E$ with the normalization $\xi(t_k) = t_k$ $(k = 0, n+1)$ and $\xi(\infty) = \infty$ that satisfies the equation of the form of (15):

$$\xi_{\bar\zeta} - \mu_1^0 \xi_\zeta - \mu_2^0 \bar z_{\bar\xi}/(z_\xi \bar\xi_{\bar\zeta}) = 0, \qquad \mu_k^0 = \mu_k[\zeta, z^0(\zeta)].$$

By the construction, $z_{\bar\xi} = 0$, i.e., $z = F[\zeta(\xi)] \equiv F_0(\xi)$, is an analytic function. Since $\zeta(\xi) \in W_{p_0}^1$ and $p_0 = p_0(m_0) > 2$, then $z = F_0(\xi)$ satisfies boundary conditions (10) with $H[t(\tau)] \equiv H_0(\tau) \in SW_{p>2}^1(l)$, $l$: $|\tau| > 1$, and $p = p(p_0, m_0)$. Then, according to Theorem 1, $z = F_0(\xi) \in W_{p>2}^1(Q_\rho)$.

Converting back to the variable $\zeta$, we obtain the following estimate for the solution of the problem (10), (15):

$$\|z(\zeta)\|_{Q_\rho}^{1,p} = C(\delta, m_0, \rho) < \infty, \qquad p > 2. \tag{16}$$

Here $\|\varphi\|_\Omega^{1,p} = \|\varphi\|_{W_p^1(\Omega)}$ and $\delta$ is a characteristic of $P \subset \partial D$.

The analysis performed in [6, 8] shows that the functions $H(t)$ in Eq. (5) and $g(x)$ in Eq. (2) have the same smoothness. Therefore, we shall formulate the necessary conditions in terms of the function $H(t)$. Using the theorem proved in [8], we arrive at the following statement.

**Theorem 2.** *Let $H(t) \in SW_{p>2}^1(l)$ and $\mu_k(\zeta, z) \in C^\alpha(E \times D_0)$ $(\alpha > 0$ $k = 1, 2)$, where $\partial D_0 = P \cup P_0 \cup P_{n+1}$, $P_j = \{z\colon x = x_j,\ y < y_j\}$ $(j = 0, n+1)$. Then, the problem* (10), (15) *for the simple polygon $P$ [Assumption* (i)*] has at least one solution $z = F(\zeta)$, $F$: $E \to D$ and the estimate* (16) *holds for this solution.*

*If $dH/dt = \Pi_0(t)h_*(t)$, $\ln h_* \in C^\alpha(l)$, and $\mu_k(\zeta, z) \in C^\alpha(E \times D_0)$ ($\alpha > 0$, $k = 1, 2$), then $\omega = z_\zeta$ satisfies the boundary-value problem* (6), *in which $t_k$ are subject to inequalities* (8), *and*

$$\|z(\zeta)\|^{1,p}_{Q_\rho} \leqslant C, \qquad \|\Pi_* z_\zeta\|^{(\nu)}_{Q_\rho} \leqslant C(\varepsilon, m_0, \rho), \quad \nu > 0, \tag{17}$$

*where the weight function $\Pi_*$ is determined after formula* (8) *is derived.*

**Remark 1.** Using the quasiconformal mapping method developed in [6, pp. 275–279], we can weaken the constraints imposed on the coefficients $\mu_k(\zeta, z)$ in the first part of Theorem 2, assuming only that they are continuous over $z$ for almost all $\zeta \in E$.

**4. Problems of Nonlinear Filtration in the Canonical Domain.** As in Sec. 3, Eq. (2) is converted to the variable $\zeta$ by the conformal mapping $w = W(\zeta)$, $W$: $E \to D^*$. This yields the nonlinear equation

$$z_{\bar\zeta} - \mu_1 z_\zeta - \mu_2 \bar z_{\bar\zeta} = 0, \qquad \|\mu\| < 1, \tag{18}$$

where $\mu_k(\zeta, z, \omega) \equiv m_k(W, z, \omega W_\zeta^{-1})(\bar W_{\bar\zeta} W_\zeta^{-1})^{2-k}$ ($k = 1, 2$, $\omega = z_\zeta$). Equation (18) for $z = F(\zeta)$, $F$: $E \to D$ is treated as the initial equation of nonlinear filtration (2) for the function $z = z(w)$ ($w \in D^*$). Therefore, the conditions imposed on the filtration tensor $K(z, \varphi, \nabla\varphi)$ in Eq. (1) are also expressed in terms of the coefficients $\mu_k$ of Eq. (18). Equation (18) is formally differentiated with respect to $\zeta$, and the resulting equation is written as

$$\omega_{\bar\zeta} - q_1 \omega_\zeta - q_2 \bar\omega_{\bar\zeta} = \sum_{k+l=0}^{2} a_{kl} \omega^k \bar\omega^l \equiv a, \qquad \|q\| < 1, \tag{19}$$

where $q = (q_1, q_2)$ and $\|q\| = \sup(|q_1| + |q_2|)$.

We note that the inequality in Eq. (19) is the ellipticity condition for the nonlinear equation (18).

Assumption (iii): $\mu_k(\zeta, z, \omega) \in C^1(E \times D_0 \times \mathbb{C})$ ($k = 1, 2$), where $\partial D_0 = P \cup P_0 \cup P_{n+1}$, and $P_j = \{z\colon x = x_j, y < y_j\}$ ($j = 0, n+1$).

An apparent corollary of Assumption (iii) is the inequality $\sup|a_{kl}| < \infty$.

For the boundedness of $\mu_{k\zeta}$, the existence of the derivative $W_{\zeta\zeta}$ is necessary. This leads to the following strengthening of Assumption (ii): $(P^*, L^*) \subset C^{2+\alpha}$ ($\alpha > 0$). In filtration problems, since the boundary $\partial D^* = P^* \cup L^*$ consists of segments of straight lines $\varphi = \mathrm{const}$ and $\psi = \mathrm{const}$, the derivative $W_{\zeta\zeta}$ is bounded if $\operatorname{Im}\zeta \geqslant 0$, except for the pre-images $\zeta = \pm 1$ of the points $(w_0, w_{n+1}) \subset P^* \cap L^*$. The singularities $\mu_{k\zeta}$ at the points $\zeta = \pm 1$ complicate insignificantly the following derivations. In order that $\mu_{k\zeta}$ be bounded at the points $\zeta = \pm 1$, it suffices, for example, to assume that $m_k(w, z_j, \sigma) = 0$ ($j = 0, n+1$). Below, for the solutions $z(\zeta)$ and $\omega(\zeta)$ of Eqs. (18) and (19), respectively, we will consider the boundary-value problem (6) with the function $dH/dt = \Pi_0(t)h_*(t)$, $\ln h_* \in C^\alpha(l)$, whose properties are defined only by the smoothness of the boundary functions $g(x)$ and $G(\varphi, \psi) = 0$ in Eq. (2).

**5. Regularization of the Problem. *A Priori* Estimates.** Let us regularize the nonlinear problem. We introduce strips $E_\nu = \{\zeta\colon -\infty < \operatorname{Re}\zeta < \infty \text{ and } 0 < \operatorname{Im}\zeta < \nu\}$ ($\nu > 0$) and construct a patch function $\chi(\zeta) \in C^3(E)$ such that $\chi(\zeta) = 0$ for $\zeta \in E_\rho$ ($\rho > 0$) and $\chi(\zeta) = 1$ for $\zeta \in E \setminus E_{2\rho}$. In Eq. (18), we set $\mu_{k\rho} = \chi\mu_k$ ($k = 1, 2$); in this case, $\mu_{k\rho} = q_{k\rho} = a_{kl\rho} = 0$ if $\zeta \in E_\rho$. In the notation, we omit the subscript $\rho$, assuming that $\mu_k = q_k = a_{kl} = 0$ for $\zeta \in E_\rho$. Substituting the arbitrary measurable functions $z(\zeta)$ and $\omega(\zeta)$ into the coefficients in (18), we construct the quasiconformal map $\xi = R(\zeta)$, $R$: $E \to E$ defined before formula (16) such that $z_{\bar\xi} = 0$ ($\xi \in E$). Since $\mu_k = 0$ in the strip $E_\rho$, the conformal map $\xi = R(\zeta)$ ($\zeta \in E_\rho$) can be analytically continued by symmetry to the strip $E_\rho^-\{\zeta\colon -\infty < \operatorname{Re}\zeta < \infty, -\rho < \operatorname{Im}\zeta < 0\}$. The constructed conformal image $\xi = R^0(\zeta)$ ($\zeta \in E\rho \cup E_\rho^-$) is analytic on the straight line $\partial E$: $\operatorname{Im}\zeta = 0$, and, hence, $\|R^0(t)\|^{(2)}_{\partial E} = C(m_0, \rho) < \infty$. Then, in the transformed boundary condition (6) for the analytic function $dz/d\xi$, $\operatorname{Im}\xi > 0$, we obtain $\|\ln h_*[t(\tau)]\|^{(\alpha)}_l = C(m_0, \rho) < \infty$ and, hence, the estimates (8) and (9) hold for $dz/d\xi$. Converting back to the variable $\zeta$, we note that for the function $z(\zeta)$, the estimates (8) for the constants $t_k$ ($k = \overline{0, n+1}$) and the estimate (9) in the strip $E_\rho$ hold and, in addition, the following inequalities are valid:

$$\sup(\|\Pi_* z_\zeta\|^{(\nu)}_{E_\rho}, \|z(\zeta)\|^{1,p}_E) \leqslant C, \qquad \nu > 0, \qquad p > 2. \tag{20}$$

We write the right side of (19) as

$$a(\zeta, z, \omega) = \sum_{k+l=0}^{2} a_{kl} \omega^k \bar\omega^l = A_0 + A_1\omega + A_2\bar\omega,$$

where $A_0 = a_{00}$, $A_1 = a_{10} + a_{20} z_\zeta + a_{11} \bar{z}_{\bar{\zeta}}$, and $A_2 = a_{01} + a_{02} \bar{z}_{\bar{\zeta}}$, and set $A_k^0(\zeta) = A_k(\zeta, z^0, \omega^0)$. Here $z^0(\zeta)$ is a function that satisfies inequalities (20) and $\omega^0(\zeta)$ is an arbitrary measurable function.

By virtue of the boundedness of $a_{kl}$ and the estimate (20), we have $\|A_k^0\|_{L_p} < \infty$, $p > 2$. We denote the resulting quasilinear equation in terms of ($19^0$) and note that for the solution $\omega = \omega^1(\zeta)$ of problem (6), ($19^0$), the following weight estimate holds [6, p. 275]:

$$\|\Pi_* \omega\|_E^{1,p} \leqslant C < \infty, \qquad p = p(m_0, q_0) > 2. \tag{21}$$

Here the weight function $\Pi_*(\zeta)$ is determined after inequalities (8), which are satisfied in this case, as proved above. Let us revert to Eq. (18), setting $\mu_k(\zeta, z^0, \omega^1) \equiv \mu_k^1(\zeta)$ and denoting the linear equation obtained above by ($18^1$). Since $\mu_k = 0$, $\zeta \in E_\rho$, and $\rho > 0$, by virtue of (21), we have

$$\|\omega\|_{E \setminus E_\rho}^{(\nu)} = C(\rho) < \infty \quad \Rightarrow \quad \|\mu_k^1\|_E^{(\nu)} = C_0(\rho) < \infty, \quad \nu > 0.$$

Then, for the solution $z = z^1(\zeta)$ of problem (6), ($18^1$), an estimate in $E \setminus E_\rho$ holds that is similar to that in Eq. (17).

Taking into account inequalities (20), we finally arrive at the following *a priori* weight estimate of the solution $z = z(\zeta)$ of the regularized problem (6), (18) ($\mu_k = 0$, $\zeta \in E_\rho$):

$$\|\Pi_* z_\zeta\|_E^{1,p} \leqslant C(m_0, \varepsilon, \rho), \qquad p > 2. \tag{22}$$

**6. Solvability of the Problem.** By virtue of the *a priori* estimates (21) and (22), the functions $u(\zeta) \equiv \Pi_* z_\zeta$ and $v(\zeta) \equiv \Pi_* \omega$ belong to a set $N$ from the space $W_{p>2}^1(E)$:

$$\{(u, v)\colon \|(u, v)\|_E^{1,p} = C(C_0, C_1),\ p > 2\} \equiv N. \tag{23}$$

We choose an arbitrary element $(u^*, v^*) \in N_0 \subset C^\beta(E)$, $\beta = (p - 2)/p$:

$$\{(u, v)\colon \|(u, v)\|_E^\beta = \bar{C}(C, p),\ \beta > 2\} \equiv N_0 \supset N; \tag{24}$$

substitute $z^* = u^* \Pi_*^{-1}$ and $\omega^* = v^* \Pi_*^{-1}$ into the coefficients of Eqs. (18) and (19), setting

$$\mu_k^*(\zeta) = \mu_k(\zeta, z^*, \omega^*), \qquad q_k^*(\zeta) = q_k(\zeta, z^*, \omega^*), \qquad a^*(\zeta) = a(\zeta, z^*, \omega^*).$$

We then denote the obtained equations by ($18^*$) and ($19^*$) and construct solutions $z = z^1(\zeta)$ and $\omega = \omega^1(\zeta)$ of problems (6), ($18^*$) and (6), ($19^*$) that satisfy inequalities (21) and (22). Therefore, the functions $u^1(\zeta) = \Pi_* z_\zeta^1$ and $v^1(\zeta) = \Pi_* \omega^1$ belong to the set $N$ specified in Eq. (23). Thus, we constructed an operator $\Lambda = (\Lambda_1, \Lambda_2)$ that associates a vector $(u^*, v^*) \in N_0 \subset C^\beta(E)$ ($\beta > 2$) with a vector $(\Lambda_1 u^*, \Lambda_2 v^*) = (u^1, v^1) \in N \subset\subset N_0$, where the sign $\subset\subset$ denotes a compact embedding. By virtue of the continuity of the coefficients of Eqs. (18) and (19) over $z$ and $\omega$, the operator $\Lambda\colon N_0 \to N \subset N_0$ is bounded, continuous, and compact on the set $N_0$ defined in (24). Therefore, according to Shauder's theorem, there exists at least one immovable transformation point $\Lambda$

$$(u, v) = \Lambda(u, v), \qquad (u, v) \in N_0 \subset C^\beta(E), \quad \beta > 0$$

that corresponds, by construction, to the solution $z = \int_{-1}^{\zeta} \Pi_*^{-1} u \, d\zeta$ of the regularized nonlinear problem (6), (18).

To derive a solution of the initial problem (10), (18) for $\rho = 0$, we substitute arbitrary measurable functions $z^0(\zeta)$, $\omega^0(\zeta)$ into the coefficients $\mu_k$ and consider the map $\xi = R^0(\zeta)$, $R^0 : E \to E$ in the same manner as in Sec. 5. Then, the analytic function $z = F^0(\xi)$, $F^0\colon E \to D$ satisfies the transformed boundary conditions (10) with $H[t(\tau)] \in SW_{p>2}^1(l)$ and the estimate (14). Converting back to the variable $\zeta$, we obtain a solution of problem (10), (18) that has the same properties. Thus, we proved the following theorem:

**Theorem 3.** *There exists at least one solution of the nonlinear regularized problem* (6), (18) ($\mu_k = 0$, $\zeta \in E_\rho$) *that satisfies inequalities* (20) *and* (22). *If* $\rho = 0$, *the solution* $z = F(\zeta)$ *of the limit nonlinear equation* (18) ($\mu_k \not\equiv 0$, $\zeta \in E_\rho$) *satisfies boundary conditions* (10) *almost everywhere and inequality* (14) *holds for it.*

**7. Hydrodynamic Analysis of Results.** The mathematical results obtained, as a rule, have not been interpreted in the context of the filtration problems considered. Here we try to fill this gap.

First of all, let us consider a hydrodynamic interpretation of the new mathematical result formulated in Theorem 1 on the solvability of problem (10) for analytic functions in the classes $C^\alpha(E)$, $\alpha > 0$, and $W_p^1(E)$ ($p > 2$). In filtration problems, the boundary $\partial D^* = P^* \cup L^*$ consists of segments of straight lines $\varphi = \mathrm{const}$ and $\psi = \mathrm{const}$, and, hence, it is a piecewise analytic curve. Therefore, in gravity filtration problems, the boundary function $H(t)$ is also analytic at the interior points $t \in l$. Another situation arises in designing hydraulic structures.

Here the pre-image $L^*$ of the free boundary $L$ is also a segment of a straight line $\varphi = \text{const}$ or $\psi = \text{const}$ but the function $w = g(x)$, $z \in L$ in (2) is defined, as a rule, as an insufficiently smooth function because $L$ can contain grooves ($\psi = \text{const}$ on $L$) or drainage slots ($\varphi = \text{const}$ on $L$) [1]. Therefore, the results of theorem 1 are important from the practical viewpoint.

Unlike in [8], where problems of fluid filtration in an inhomogeneous soil were studied, in Sec. 3 of the present paper, we studied the properties of a complex potential up to the boundary of the filtration domain. We note, in particular, the important fact of boundedness of the free boundary $L$ proved in Theorem 2 [estimate (14)].

The main results of this study are reported in Secs. 4–6, focused on fluid filtration in nonideal porous media (inhomogeneous or anisotropic with a nonlinear resistance law). Since the filtration problems are substantially nonlinear in this case, we proved the existence of only a generalized solution of the nonlinear filtration equation that satisfies the boundary-value problem (10) almost everywhere. In this case, if the soil is homogeneous in a small neighborhood of the filtration domain $D$ and Darcy's law is linear, which corresponds to a rather large time of filtration flow through the domain $D$, then, according to the first part of Theorem 3, the solutions of the nonlinear filtration problem have the same properties as in the case of an ideal porous medium described by analytic functions (see Sec. 2).

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